Row and Column Spaces of a Matrix

Let A be an $m \times n$ matrix.

Defn: The row space of A is the vector space spanned by the rows of A. We denote this space by $\text{row}(A)$. The row-rank of A is $\dim(\text{row}(A))$.

Ex: Let $M = \begin{bmatrix} 3 & 2 & 8 & -1 & 0 \\ 1 & 7 & 6 & 1 & 1 \\ 4 & 1 & 7 & 0 & -5 \end{bmatrix}$ ← 3×5 matrix.

$$\text{row}(M) = \text{span}\left\{ \begin{matrix} [3\ 2\ 8\ -1\ 0], \\ [1\ 7\ 6\ 1\ 1], \\ [4\ 1\ 7\ 0\ -5] \end{matrix} \right\} \leq M_{1,5}(\mathbb{R}).$$

What is row-rank of M? Want: basis!

$$\begin{bmatrix} 3 & 2 & 8 & -1 & 0 \\ 1 & 7 & 6 & 1 & 1 \\ 4 & 1 & 7 & 0 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 7 & 6 & 1 & 1 \\ 3 & 2 & 8 & -1 & 0 \\ 4 & 1 & 7 & 0 & -5 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{r} \ell_1 = \\ \rightsquigarrow \ell_2 = \\ \ell_3 = \end{array} \begin{bmatrix} 1 & 7 & 6 & 1 & 1 \\ 0 & -19 & -10 & -4 & -3 \\ 0 & -27 & -17 & -4 & -9 \end{bmatrix}$$

← observe: last 2 rows are lin indep of one another (not scalar multiples...)

Moreover, $\{\ell_1, \ell_2, \ell_3\}$ is lin indep.

So row-rank of M is 3. $\boxtimes$

Prop: Suppose A is a matrix. The row space of A has basis the nonzero rows of $RREF(A)$. $\boxtimes$

$\hookrightarrow$ A is row-equiv to $RREF(A)$, so
$row(A) = row(RREF(A))$...

Point: To compute a basis of $row(A)$, compute $RREF(A)$ and use the nonzero rows :).

Cor: The row-rank of A is the number of leading 1's in $RREF(A)$.

pf: # leading 1's in $RREF(A)$ = # nonzero rows $RREF(A)$

Def^n: The <u>column space</u> of A is the span of the columns of A. We denote this by $col(A)$. The column-rank of A is $\dim(col(A))$.

Ex: Let $M = \begin{bmatrix} 1 & 3 & 5 & 0 & -2 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & -5 & -10 & 1 & 4 \end{bmatrix}$.

To compute the column space:

$$col(M) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -10 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \right\}$$

Use $RREF(M)$!

$$\begin{bmatrix} 1 & 3 & 5 & 0 & -2 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & -5 & -10 & 1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 5 & 0 & -2 \\ 0 & -5 & -10 & 1 & 4 \\ 0 & -5 & -10 & 1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 5 & 0 & -2 \\ 0 & 5 & 10 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 3 & 5 & 0 & -2 \\ 0 & 1 & 2 & -\frac{1}{5} & -\frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 3/5 & -2/5 \\ 0 & 1 & 2 & -\frac{1}{5} & -\frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When we choose a subset of the columns of M and ask about lin. ind., we get a 0-row for any 3 ...

Trying those:

3×2 system $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 0 & -5 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Interpretation: The first 2 vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}\begin{bmatrix}3\\1\\-5\end{bmatrix}$ are L.I.

Hence: $\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\-5\end{bmatrix} \right\}$ is a basis of $\mathrm{Col}(A)$.

$\therefore$ the column-rank of A is 2.

NB: Row-rank of this A is also 2 ... ☒

Prop: Let A be an $m \times n$ matrix. The column space of A has basis

$$B = \left\{ v_i : \begin{array}{l} v_i \text{ is the } i^{th} \text{ column of } A, \\ \mathrm{RREF}(A) \text{ has a leading 1 in column } i \end{array} \right\}.$$ ☒

Cor: The column-rank of A is the number of leading 1's in $\mathrm{RREF}(A)$. ☒

Cor: The row-rank of A is the same as the column-rank of A. ☒

pf: We gave them the same description!

Defn: The rank of A is $\mathrm{rank}(A) = \dim(\mathrm{row}(A)) = \dim(\mathrm{Col}(A))$.

Def^n: The *transpose* of matrix A is the matrix A^T obtained by turning the i^{th} column of A into the i^{th} row of A^T. I.e.

for $A = [a_{i,j}]_{i,j=1}^{m,n}$ we have $A^T = [a_{j,i}]_{j,i=1}^{n,m}$.

Ex: $M = \begin{bmatrix} 1 & 0 & 1 & 5 & 5 \\ 0 & 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$, $M^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 5 & 1 & 0 \\ 5 & 5 & 0 \end{bmatrix}$.

Observation: ① $\text{row}(A) = \text{col}(A^T)$

i.e. $\text{row}(A^T) = \text{col}(A)$.

② $(A^T)^T = A^{TT} = A$.

Cor: For all matrices A, $\text{rank}(A) = \text{rank}(A^T)$.

pf: $\text{rank}(A) = \dim(\text{col}(A))$

$= \dim(\text{row}(A^T))$

$= \text{rank}(A^T)$. $\blacksquare$

Recall: Given matrix A, there is a corresponding linear transformation $L_A : \mathbb{R}^n \to \mathbb{R}^m$ for A an $m \times n$ matrix. $L_A(\vec{x}) = A\vec{x}$.

Earlier we defined: $\text{col}(A) = \text{span}\{\text{columns of } A\}$

$\overset{*}{=} \text{ran}(L_A)$

Cor: $Col(A) = ran(L_A)$ and so

$rank(A) = dim(col(A)) = dim(ran(L_A))$.

So we can define $rank(L_A) = rank(A)$.

Even better: $rank(L_A) = dim(ran(L_A))$

A: $m \times n$ matrix $= n - nullity(L_A)$.

$= n - dim(null(A))$.

where $null(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$. *

Let A be $m \times n$. $L_A : \mathbb{R}^n \to \mathbb{R}^m$.

A^T is $n \times m$. So $L_{A^T} : \mathbb{R}^m \to \mathbb{R}^n$,

but $rank(L_A) = rank(L_{A^T})$ ...

Prop: If A is an $n \times n$ matrix, the following are equivalent:

1. $rank(A) = n$.
2. $A\vec{x} = 0$ has a unique solution. *
3. A is nonsingular.
4. the rows of A span $M_{1,n}(\mathbb{R})$
5. the rows of A are lin. indep.

A is said to be non-singular.

$rank(A) = n \to dim(null(A)) = n - n = 0$

$rank(A) = n \to dim(\underbrace{row(A)}_{M_{1,n}(\mathbb{R})}) = n \to$ rows are a basis of $M_{1,n}(\mathbb{R})$.

rows (A) are lin. indep: n rows $\leadsto$ $dim(row(A)) \geq n$.

(6) the columns of A span $M_{n,1}(\mathbb{R})$.

(7) the columns of A are lin. indep.

(8) Every linear system w/ coeff matrix A has a unique solution.

Remark: (8) $\leftrightarrow$ (2) because: (8) $\to$ (2) trivial.

(2) $\to$ (8) follows from (2) $\to$ col(A) has full dimension.

$\to$ columns of A are a basis of $\mathbb{R}^n$

$\to$ (8) by unique linear combinations

B a basis of V.

$W \leq V$. $B \subseteq W$.

Then
$$\underbrace{\text{span}(B)}_{= V} \leq \underbrace{\text{span}(W)}_{= W} \leq V$$

Hence $V \leq W \leq V \Rightarrow V = W$.

$\text{ran}(L) \leq V$ when $L : W \to V$.

$B \subseteq \text{ran}(L) \Rightarrow V = \text{span}(B) \leq \text{ran}(L)$